There is an upper limit to the values of u_n which satisfy this inequality, this being the larger of the roots of the equation:

$$u_n^2 - u_n - a = 0.$$

Thus u_n cannot exceed a given quantity independent of n, and on the other hand u_n constantly increases with n. Hence, u_n *has a limit*, according to the fundamental principle of the theory of limits. Let this limit be denoted by s. The value of s may be found by passing to the limit in (2).

$$\lim_{n\to\infty} u_{n+1}^2 = \lim_{n\to\infty} u_n + a, \quad \text{or} \quad s^2 = s + a,$$

hence $s = \dfrac{1 + \sqrt{1 + 4a}}{2}$, since s is necessarily positive.

The problem may be further generalized by considering the expression

$$\sqrt[p]{a + \sqrt[p]{a + \sqrt[p]{a + \sqrt[p]{a + \cdots}}}} \text{ to infinity}$$

where p is a positive integer, and a being any real quantity when p is odd, or any positive quantity, when p is even. The method of solution would be the same, in the main, as the one outlined above.

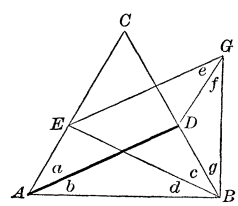
Also solved by W. P. Ransom, C. N. Schmall, A. M. Harding, E. E. Clark, Horace Olson, O. S. Adams, H. N. Carleton, Paul Capron, J. M. Stetson, N. P. Pandya, and the Proposer.

GEOMETRY.

487. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If segments from the vertices A and B of a triangle to the opposite sides are of equal length and divide the angles A and B proportionally, the triangle is isosceles.

I. Solution by J. J. Ginsburg, Student, Cooper Union, New York City.



In the triangle ABC, we have by hypothesis $AD = BE$ and

(1) $$\frac{a}{b} = \frac{c}{d} = \frac{m}{n},$$

when a, b, c, d are the measures of the angles into which $\angle A$ and $\angle B$ are divided by AD and BE, and m/n is any given ratio.

We are to prove that $\angle A = \angle B$. Suppose that $\angle A \neq \angle B$, and suppose $\angle A > \angle B$. Then from (1)

(2) $$a > c \text{ and } b > d.$$

In the triangles ABD and AEB, $AD = BE$, $AB = AB$ and $b > d$. Hence

(3) $$BD > AE.$$

Draw $EG \parallel AD$ and $DG \parallel AE$, and draw BG. Then $EG = AD$ and $AD = BE$ by hypothesis. Hence $\triangle ABG$ is isosceles.

We have, then, $e + f = c + g$ and $e = a > c$. Hence,

(4) $$f < g \quad \text{and} \quad BD < DG < AE.$$

But (3) and (4) are contradictory. Hence the supposition that $\angle A > \angle B$ is impossible; Likewise $\angle A < \angle B$ is impossible. Hence $\angle A = \angle B$, and triangle ABC is isosceles.

II. Solution by Paul Capron, U. S. Naval Academy.

Denote the length of each segment by l, and the length of $\overline{AB}$ by c. Let the part of the angle A next to c be kA; then the part of the angle B next to c is kB. $[0 < k < 1.]$

Then

$$\frac{l}{\sin B} = \frac{c}{\sin (kA + B)}, \quad \text{and} \quad \frac{l}{\sin A} = \frac{c}{\sin (kB + A)}.$$

Hence,

$$\frac{\sin B}{\sin A} = \frac{\sin (kA + B)}{\sin (kB + A)}.$$

By composition and division,

$$\frac{\sin B - \sin A}{\sin B + \sin A} = \frac{\sin (kA + B) - \sin (kB + A)}{\sin (kA + B) + \sin (kB + A)};$$

whence,

$$\frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)} = \frac{\tan \dfrac{1-k}{2}(B - A)}{\tan \dfrac{1+k}{2}(B + A)}; \quad \text{or} \quad \frac{\tan \frac{1}{2}(B - A)}{\tan \dfrac{1-k}{2}(B - A)} = \frac{\tan \frac{1}{2}(B + A)}{\tan \dfrac{1+k}{2}(B + A)}$$

Since $0 < k < 1$ and the tangent is an increasing function, it follows that if B and A were unequal, the last relation would be untrue, for then the left-hand fraction would be greater than unity, and the right-hand fraction would be less than unity. Hence, $B = A$, and the triangle is isosceles.

488. Proposed by ROGER A. JOHNSON, Western Reserve University.

If triangles are constructed on a given base, having the radii of the incircle and circumcircle in a constant ratio, determine the locus of the vertex (necessarily the constant ratio is not greater than $\frac{1}{2}$).

Solution by Paul Capron, U. S. Naval Academy.

Let the constant ratio be $n = 2m$, the given base a, with its left end at B, its right end at C; and let the sides a, b, c of any one of the triangles be opposite the vertices A, B, C respectively.

Then

$$n = 2m = = \frac{ac \sin B}{a + b + c} \div \frac{b}{2 \sin B} = \frac{2ac \sin^2 B}{b(a + b + c)}.$$

Let $c = r$, $B = \theta = 2\phi$; then $b^2 = a^2 + r^2 - 2ar \cos \theta = (a + r)^2 - 4ar \cos^2 \phi$. On substituting these values, we have, for the locus of A:

(1) $$m(a + r) \sqrt{(a + r)^2 - 4ar \cos^2 \phi} = 4ar \cos^2 \phi(m + \sin^2 \phi) - m(a + r)^2.$$